

August 2003, Day 1, Question 1

For what values of α and β does the series

$$\sum_{n=2}^{\infty} \frac{1}{n^{\alpha} (\log(\log n))^{\beta} \log n}$$

converge? In this problem all logarithms are taken base 2, *i.e.* $\log x = \log_2 x$.

To solve this we appeal to the following fact, found as Theorem 3.27 in *Principles of Mathematical Analysis* by Walter Rudin:

Suppose $a_1 \geq a_2 \geq a_3 \geq \dots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \dots$$

converges.

The terms of the given series satisfy the assumptions of this theorem. In this case,

$$\begin{aligned} \sum_{k=2}^{\infty} 2^k a_{2^k} &= \sum_{k=2}^{\infty} \frac{2^k}{2^{\alpha k} (\log(\log 2^k))^{\beta} \log 2^k} \\ &= \sum_{k=2}^{\infty} \frac{2^{(1-\alpha)k}}{(\log k)^{\beta} k}. \end{aligned}$$

If $\alpha < 1$, then $1 - \alpha > 0$, $2^{1-\alpha} > 1$, and asymptotically $2^{(1-\alpha)k}$ grows faster than $(\log k)^{\beta} k$ for any value of β . Thus for $\alpha < 1$, this series diverges regardless of the value of β .

On the other hand, if $\alpha > 1$, then $1 - \alpha < 0$, and $2^{1-\alpha} < 1$, so $2^{(1-\alpha)k}$ decays rapidly. In this case,

$$\begin{aligned} \sum_{k=2}^{\infty} \frac{2^{(1-\alpha)k}}{(\log k)^{\beta} k} &\leq \frac{1}{(\log 2)^{\beta} 2} \sum_{k=2}^{\infty} 2^{(1-\alpha)k} \\ &= \frac{1}{2} \sum_{k=2}^{\infty} (2^{1-\alpha})^k \\ &< \frac{1}{2} \sum_{k=0}^{\infty} (2^{1-\alpha})^k \\ &= \frac{1}{2} \left(\frac{1}{1 - 2^{1-\alpha}} \right) < \infty, \end{aligned}$$

where we have used the fact that

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \quad \text{for } |z| < 1.$$

Note that β played no rôle here. Hence if $\alpha > 1$ the series converges regardless of the value of β .

Now we must consider $\alpha = 1$. If $\alpha = 1$ we have

$$\sum_{k=2}^{\infty} \frac{1}{(\log k)^{\beta} k}.$$

To determine appropriate values of β , we appeal to Theorem 3.27 again:

$$\begin{aligned} \sum_{k=2}^{\infty} 2^k b_{2^k} &= \sum_{k=2}^{\infty} \frac{2^k}{(\log 2^k)^{\beta} 2^k} \\ &= \sum_{k=2}^{\infty} \frac{1}{k^{\beta}}. \end{aligned}$$

The series in the last line converges if $\beta > 1$ and diverges if $\beta \leq 1$. Hence, if $\alpha = 1$, the original series converges if $\beta > 1$ and diverges if $\beta \leq 1$.